DETERMINING THE HEAT CONDUCTION OF ANISOTROPIC MEDIA ON THE BASIS OF THE SCANNING METHOD. THEORETICAL BASIS OF THE METHOD

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The solution of the heat-conduction equation is obtained for an anisotropic semiinfinite medium heated by a mobile heat source. The temperature fields of various types of mobile source are analyzed, with a view to developing a method of determining the heat conduction of anisotropic media.

One means of nondestructive determination of the thermal properties of materials is to use a measurement method based on mobile sources of thermal energy — the scanning method [1]. The introduction of mobile sources of thermal energy into the practice of thermophysical measurements and the use of modern means of heating and temperature recording permits the creation of a set of different contactless procedures for determining the thermal properties of materials (including hot rocks) on the basis of the scanning method [1, 2].

The study of the anisotropy in the thermal properties of materials occupies a special position here. It is known that traditional methods and means of measurement used previously to determine the heat conduction of anisotropic minerals [3-5] do not offer the possibility of large-scale measurement, because of their inadequate productivity, as well as the need for preliminary mechanical treatment of the samples (for example, single crystals of minerals), which leads to partial or total destruction of the samples. Accordingly, there is a pressing need to create effective methods for the contactless determination of the heat conduction of anisotropic methods without rigid requirements on their geometric shape and size.

Consider a semiinfinite anisotropic solid medium with an adiabatic boundary surface, on which an arbitrary mobile source of thermal energy acts; the source moves over the boundary surface at constant velocity **v**. The general solution of the differential heat-conduction equation for the given case may be obtained using Green's functions [6]. Introducing the mobile coordinate system OX'Y'Z', with its origin at the instantaneous position of the point source of thermal energy and the axes X', Y', Z' along the principal axes of the heat conduction of the medium (Fig. 1), the equation for the Green's function G(x', y', z', t) in this system takes the form

$$\frac{\partial G}{\partial t}=\frac{1}{c\rho}\left(\lambda_1\frac{\partial^2 G}{\partial (x')^2}+\lambda_2\frac{\partial^2 G}{\partial (y')^2}+\lambda_3\frac{\partial^2 G}{\partial (z')^2}\right)+\mathbf{v}\nabla G+\frac{2Q}{c\rho}\,\delta(t)\,\delta(x')\,\delta(y')\,\delta(z'), \tag{1}$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$ are the principal thermal conductivities of the medium in the direction of the axes X', Y', Z', respectively; Q is the energy of the instantaneous source. The factor 2 in the last term on the right-hand side of Eq. (1) takes account of the adiabaticity of the boundary surface of the medium.

Solving Eq. (1), the Green's function in the coordinate system OX'Y'Z' is obtained:

$$G(x',\,y',\,z',\,t)=\frac{2Q\,(c\rho)^{1/2}}{(4\pi t)^{3/2}\,(\lambda_1\lambda_2\lambda_3)^{1/2}}\exp\left\{-\frac{c\rho}{4t}\left[\frac{(x')^2}{\lambda_1}+\frac{(y')^2}{\lambda_2}+\right.\right.$$
$$\left.\left.+\frac{(z')^2}{\lambda_3}\right]-\frac{c\rho v}{2}\left(\frac{n_{x'}x'}{\lambda_1}+\frac{n_{y'}y'}{\lambda_2}+\frac{n_{z'}z'}{\lambda_3}\right)-\frac{c\rho v^2 t}{4}\left(\frac{n_{x'}^2}{\lambda_1}+\frac{n_{y'}^2}{\lambda_2}+\frac{n_{z'}^2}{\lambda_3}\right)\right\}, \tag{2}$$

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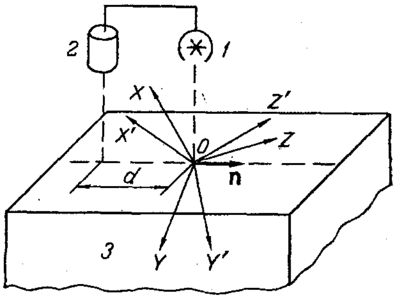


Fig. 1

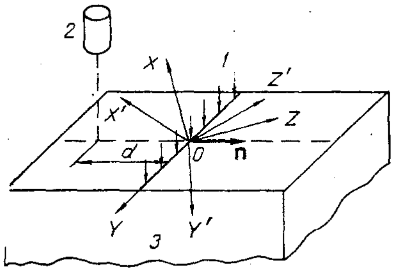


Fig. 2

Fig. 1. Semiinfinite anisotropic medium heated by a mobile point energy source: 1) energy source; 2) temperature recording element; 3) anisotropic medium; **n**, direction of motion of energy source and temperature recording element.

Fig. 2. Heating a semiinfinite anisotropic medium with a mobile linear source of thermal energy. Notation as in Fig. 1.

where $n_{x'} = v_{x'}/v$, $n_{y'} = v_{y'}/v$, $n_{z'} = v_{z'}/v$ are the directional cosines of the unit vector $\mathbf{n} = \mathbf{v}/v$.

For practical applications, it is of interest to have an expression for the Green's function in the mobile coordinate system OXYZ moving with the point energy source, which is arbitrarily oriented relative to the principal axes of heat conduction (Fig. 1). Conversion to the coordinate system OXYZ gives

$$G(x,\ y,\ z,\ t) = \frac{2Q\,(c\rho)^{1/2}}{(4\pi t)^{3/2}\,(\lambda_1\lambda_2\lambda_3)^{1/2}} \exp\left[-c\rho\left(M + \frac{L}{4t} + Nt\right)\right], \tag{3}$$

where

$$\begin{gathered} L = \frac{(\mathbf{ri}')^2}{\lambda_1} + \frac{(\mathbf{rj}')^2}{\lambda_2} + \frac{(\mathbf{rk}')^2}{\lambda_3}, \\ M = v\,\frac{(\mathbf{ri}')(\mathbf{ni}')}{2\lambda_1} + v\,\frac{(\mathbf{rj}')(\mathbf{nj}')}{2\lambda_2} + v\,\frac{(\mathbf{rk}')(\mathbf{nk}')}{2\lambda_3}, \\ N = v^2\,\frac{(\mathbf{ni}')^2}{4\lambda_1} + v^2\,\frac{(\mathbf{nj}')^2}{4\lambda_2} + v^2\,\frac{(\mathbf{nk}')^2}{4\lambda_3}, \end{gathered} \tag{4}$$

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the radius vector of an arbitrary point of the medium in the coordinate system OXYZ. The coordinates (x, y, z) of vector **r** in the coordinate system OXYZ are related as follows to its coordinates (x', y', z') in the coordinate system OX'Y'Z' [7]:

$$\begin{aligned} x &= \alpha_x x' + \beta_x y' + \gamma_x z', \\ y &= \alpha_y x' + \beta_y y' + \gamma_y z', \\ z &= \alpha_z x' + \beta_z y' + \gamma_z z', \end{aligned} \tag{5}$$

where $\alpha_m$, $\beta_m$, $\gamma_m$ (m = x, y, z) are the directional cosines of the axes X, Y, Z relative to the principal axes of heat conduction of the medium X', Y', Z'.

In heating the surface of the given medium by an arbitrary mobile source of thermal energy with a power density q(x, y, z, t), the temperature field of the medium is given by the relation [6]

$$\theta(x,\ y,\ z,\ t) = \int_{-\infty}^{t}\int_{S} q(\tilde{x},\ \tilde{y},\ \tilde{z},\ \tilde{t})\,G(x-\tilde{x},\ y-\tilde{y},\ z-\tilde{z},\ t-\tilde{t}) \tag{6}$$

where S is the geometric focus of the points at which heat liberation occurs (the area of the energy source); S is determined by the type of surface source.

Using Eqs. (3)-(6), the temperature fields arising in a semiinfinite anisotropic medium on heating by mobile energy sources of the types most often encountered in thermophysical investigation — continuously acting point and linear sources [1] — may be considered.

Continuously Acting Point Source of Thermal Energy

Suppose that at time t = 0 a point source of thermal energy with constant power W begins to act on the surface of a semiinfinite anisotropic medium. In the coordinate system OXYZ (Fig. 1), in accordance with Eqs. (3)-(6), the temperature field of the medium is

$$\theta(x,\ y,\ z,\ t)=\frac{w}{\pi^{3/2}(\lambda_1\lambda_2\lambda_3)^{1/2}L^{1/2}}\exp(-c\rho M),$$
$$\int\limits_{\frac{1}{2}(c\rho L/t)^{1/2}}^{\infty}\exp(-\xi^2-k^2\xi^{-2})\,d\xi, \tag{7}$$

where

$$\xi^2=\frac{c\rho L}{4}(t-\tilde{t})^{-1},\qquad k=\frac{1}{2}c\rho(LN)^{1/2}. \tag{8}$$

In the practical use of mobile energy sources in thermophysical investigations, heating conditions described by the quasi-steady approximation are most widespread [1, 2]; this permits significant simplification in the relations for the temperature field. Passing to the limit as $t\to\infty$ in Eq. (7), the relation for the steady field of limiting excess temperature in quasi-steady heating conditions is obtained for a semiinfinite anisotropic medium in the mobile coordinate system OXYZ

$$\theta(x,\ y,\ z)=\frac{w}{2\pi(\lambda_1\lambda_2\lambda_3)^{1/2}L^{1/2}}\exp[-c\rho M-c\rho(LN)^{1/2}]. \tag{9}$$

In the experimental realization of the methods of investigating the heat conduction of anisotropic solids described below, interest centers on the distribution of the limiting excess temperature at the line of heating of the boundary surface of the medium behind the energy source relative to its direction of motion (Fig. 1). This distribution, according to Eq. (9), is determined by the formula

$$\theta(x,\ y,\ z)=\frac{w}{2\pi(\lambda_1\lambda_2\lambda_3)^{1/2}L^{1/2}}=\frac{w}{2\pi d(\lambda_1\lambda_2\lambda_3)^{1/2}}\left[\frac{(\mathbf{n}\mathbf{i}')^2}{\lambda_1}+\frac{(\mathbf{n}\mathbf{j}')^2}{\lambda_2}+\frac{(\mathbf{n}\mathbf{k}')^2}{\lambda_3}\right]^{-1/2}, \tag{10}$$

where d is the distance from the point energy source to the point of temperature recording on the line of heating behind the source with coordinates $x=-n_x d$, $y=-n_y d$, $z=-n_z d$. Analysis of Eq. (9) shows that the given temperature field (at specified w, $\lambda_1$, $\lambda_2$, $\lambda_3$) is determined only by the direction $\mathbf{n}$ of energy source motion and the distance d from the source to the point of temperature recording and, as in the case of heating an isotropic medium, does not depend on the source velocity [8].

In the particular case when the direction of motion of the point energy source coincides with one of the axes of the mobile coordinate system OXYZ, for example, with the X axis (when y = z = 0, d = $|x|$, $n_x = 1$, $n_y = n_z = 0$), Eq. (10) takes the form

$$\theta(x)=\frac{w}{2\pi(\lambda_1\lambda_2\lambda_3)^{1/2}\left(\frac{\alpha_x^2}{\lambda_1}+\frac{\beta_x^2}{\lambda_2}+\frac{\gamma_x^2}{\lambda_3}\right)^{1/2}|x|}. \tag{11}$$

If the direction of energy-source motion coincides with one of the principal axes of heat conduction of the medium, for example, with the X' axis, however, the distribution of limiting excess temperatures of the boundary surface of the medium at the heating line behind the energy source is described by the simplest expression [9]

$$\theta(x') = \frac{w}{2\pi(\lambda_2\lambda_3)^{1/2}\,|x'|}. \quad (12)$$

When $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, Eq. (12) coincides with the expression for the distribution of linear excess temperatures of the boundary surface of a semiinfinite isotropic medium at the heating line behind the mobile point energy source [8].

Continuously Acting Linear Source of Thermal Energy

Consider the temperature field of a semiinfinite anisotropic medium, at the surface of which there acts a mobile linear source of thermal energy with constant linear power density q switched on at time t = 0.

Suppose that axis Y of the mobile coordinate system OXYZ coincides with the linear energy source moving over the boundary surface in an arbitrary direction **n** at a constant velocity v (Fig. 2). In quasisteady heating conditions ($t \to \infty$), an expression for the field of limiting excess temperatures of the heated medium follows from Eqs. (3)-(5)

$$\theta(x, z) = \frac{q}{2\pi(\lambda_1\lambda_2\lambda_3)^{1/2}} \int_{-\infty}^{+\infty} \frac{\exp(-c\rho E)}{(Ay^2 + By + C)^{1/2}} \exp\{-c\rho\,[Dy + N^{1/2}(Ay^2 + By + C)^{1/2}]\}\,dy, \quad (13)$$

where

$$A = \frac{\alpha_y^2}{\lambda_1} + \frac{\beta_y^2}{\lambda_2} + \frac{\gamma_y^2}{\lambda_3};$$

$$B = \frac{2\alpha_y}{\lambda_1}(x\alpha_x + z\alpha_z) + \frac{2\beta_y}{\lambda_2}(x\beta_x + z\beta_z) + \frac{2\gamma_y}{\lambda_3}(x\gamma_x + z\gamma_z);$$

$$C = \frac{1}{\lambda_1}(x\alpha_x + z\alpha_z)^2 + \frac{1}{\lambda_2}(x\beta_x + z\beta_z)^2 + \frac{1}{\lambda_3}(x\gamma_x + z\gamma_z)^2;$$

$$D = \frac{v\alpha_y}{2\lambda_1}(\mathbf{ni}') + \frac{v\beta_y}{2\lambda_2}(\mathbf{nj}') + \frac{v\gamma_y}{2\lambda_3}(\mathbf{nk}');$$

$$E = \frac{v}{2\lambda_1}(x\alpha_x + z\alpha_z)(\mathbf{ni}') + \frac{v}{2\lambda_2}(x\beta_x + z\beta_z)(\mathbf{nj}') + \frac{v}{2\lambda_3}(x\gamma_x + z\gamma_z)(\mathbf{nk}'). \quad (14)$$

Integrating Eq. (13) and using the tabular integral [10]

$$\int_{-\infty}^{+\infty} \frac{du}{(u^2 + a^2)^{1/2}} \exp[-bu - g(u^2 + a^2)^{1/2}] = 2K_0[a(g^2 - b^2)^{1/2}], \quad (15)$$

where $K_0[a(g^2 - b^2)^{1/2}]$ is a MacDonald function of the argument $a(g^2 - b^2)^{1/2}$, $a = [(C/A) - (B/2A)^2]^{1/2}$, $b = c\rho D$, $g = c\rho(NA)^{1/2}$, a final expression is obtained for the field of limiting excess temperatures of a semiinfinite anisotropic medium at whose surface there acts a continuous mobile linear source of thermal energy

$$\theta(x, z) = \frac{q}{\pi(\lambda_1\lambda_2\lambda_3)^{1/2}A^{1/2}} \exp\left(-c\rho E + \frac{c\rho BD}{2A}\right) K_0\left\{c\rho\left[\frac{C}{A} - \left(\frac{B}{2A}\right)^2\right]^{1/2}(NA - D^2)^{1/2}\right\}. \quad (16)$$

If the direction of motion of the energy source coincides with one axis of the coordinate system OXYZ, for example, the X axis (in this case, the boundary surface lies in the plane z = 0), the field of limiting excess temperatures of the heated surface is described by the relation

$$\theta(x) = \frac{q}{\pi(\lambda_1\lambda_2\lambda_3)^{1/2}A^{1/2}} \exp\left\{-\frac{c\rho v x}{2A}\left[\frac{(\alpha_x\beta_y - \alpha_y\beta_x)^2}{\lambda_1\lambda_2} + \frac{(\alpha_x\gamma_y - \alpha_y\gamma_x)^2}{\lambda_1\lambda_3} + \frac{(\beta_x\gamma_y - \beta_y\gamma_x)^2}{\lambda_2\lambda_3}\right]\right\} K_0\left\{\frac{c\rho|x|v}{2A}\left[\frac{(\alpha_x\beta_y - \alpha_y\beta_x)^2}{\lambda_1\lambda_2} + \frac{(\alpha_x\gamma_y - \alpha_y\gamma_x)^2}{\lambda_1\lambda_3} + \frac{(\beta_x\gamma_y - \beta_y\gamma_x)^2}{\lambda_2\lambda_3}\right]\right\}. \quad (17)$$

When the axes X, Y, Z of the mobile coordinate system coincide with the principal axes of heat conduction of the medium X', Y', Z', Eq. (17) takes its simplest form

$$\theta(x') = \frac{q}{\pi(\lambda_1\lambda_3)^{1/2}} \exp\left(-\frac{c\rho x' v}{2\lambda_1}\right) K_0\left(\frac{c\rho |x'| v}{2\lambda_1}\right), \tag{18}$$

coinciding, when $\lambda_1 = \lambda_3 = \lambda$, with the expression for the field of limiting excess temperatures of the boundary surface of a semiinfinite isotropic medium on which there acts a mobile linear source of thermal energy [8]. Using the well-known approximation [7] of the MacDonald function $K_0(u)$ at large values of the argument (with an error of no more than 2% when $u > 5$), a relation that is convenient in practical calculations is obtained for the temperature field of the surface of the given anisotropic medium behind the linear energy source, i.e., when $x' < 0$, $z = 0$:

$$\theta(x') = q(\pi\lambda_3 c\rho v |x'|)^{-1/2}. \tag{19}$$

Using the relations obtained for the temperature fields of the mobile energy sources in anisotropic media, a series of methods may be developed for investigating the heat conduction of anisotropic media, for example, single crystals of minerals and rocks. Their theoretical models will be considered in a separate work.

## NOTATION

G, Green's function; $\lambda_1$, $\lambda_2$, $\lambda_3$, principal thermal conductivities of the anisotropic medium; **i**, **j**, **k**, basis vectors of the arbitrary rectangular coordinate system OXYZ; **i'**, **j'**, **k'**, basis vectors of the coordinate system OX'Y'Z', the axes of which coincide with the principal axes of heat conduction of the anisotropic medium; $\delta(t)$, Dirac delta function; $c\rho$, volumetric specific heat; $\mathbf{n} = \mathbf{v}/v$, unit vector of the scanning direction; v, source velocity; Q, source energy; w, source power; q, power density of source; $\theta(x, y, z)$, limiting excess temperature of medium; $K_0(u)$, MacDonald function; d, distance from source to temperature recording unit; $\alpha$, $\beta$, $\gamma$, directional cosines of vector in coordinate system OX'Y'Z'.

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